

‘Circles in the Sky’ in twisted cylinders

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Abstract

It is shown here how prior estimates on the local shape of the universe can be used to reduce, to a small region, the full parameter space for the search of circles in the sky. This is the first step towards the development of efficient strategies to look for these matched circles in order to detect a possible nontrivial topology of our Universe. It is shown how to calculate the unique point, in the parameter space, representing a pair of matched circles corresponding to a given isometry g (and its inverse). As a consequence, (i) given some fine estimates of the covering group Γ of the spatial section of our universe, it is possible to confine, in a very effective way, the region of the parameter space in which to perform the searches for matched circles, and reciprocally (ii) once identified such pairs of matched circles, one could determine with greater precision the topology of our Universe and our location within it.

It has recently been suggested that the quadrupole and octopole moments of the CMB anisotropies are almost aligned, i.e. each multipole has a preferred axis along which power is suppressed and both axes almost coincide. In fact, the angle between the preferred directions of these lowest multipoles is $\sim 10^\circ$, while the probability of this occurrence for two randomly oriented axes is roughly $1/62$. There is also at present almost no doubt that the extremely low value of the CMB quadrupole is a real effect, i.e. it is not an illusion created by foregrounds [1].

Traditionally, the low value of the quadrupole moment has been considered as indirect evidence for a non-trivial topology of the universe. Actually, it was the fitting to these low values of the quadrupole and octopole moments of the CMB anisotropy which motivated the recent proposal that our Universe would be a Poincaré’s dodecahedron [2]. On the other hand, the observed alignment of the quadrupole and the octopole moments has recently been used as a hint for determining the direction along which might occur the shortest closed geodesics characteristic of multiply connected spaces [3].

However, in most of the studies reported, the model topology used for the comparison with data has been the T^1 topology, i.e. the torus topology with one scale of compactification

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of the order of the horizon radius, and the other two much larger. This is the simplest topology after the trivial one. Tests using S -statistics [4] and the *circles in the sky* method [5] performed in [3] yielded a null result for a non-trivial topology of our universe. However it should be reminded that multiply connected universe models cannot be ruled out on these grounds. In fact, S -statistics is a method sensitive only to translational isometries, while the search for the *circles in the sky*, which in principle is sensitive to detect any topology, was performed in a *three-parameter version* able to detect translations only.

If the topology of the Universe is detectable in the sense of [6], then CMB anisotropy maps might present matched circles, i.e. pairs of circles along of which the anisotropy patterns match [5]. These circles are actually the intersections (in the universal covering space of the spatial sections of spacetime) of the topological images of the sphere of last scattering, and hence are related by the isometries of the covering group Γ . Since matched circles will exist in CMB anisotropy maps of any universe with a detectable topology, i.e. regardless of its geometry and topology, it seems that the search for ‘circles in the sky’ might be performed without any *a priori* information of what the geometry and topology of the universe is. However, any pair of matched circles is described as a point in a six-dimensional parameter space, which makes a full-parameter search computationally expensive.¹

Nevertheless, such a titanic search is currently being performed, and preliminary results have shown the lack of antipodal, and approximately antipodal, matched circles with radii larger than 25° [7]. These results rule out the Poincaré’s dodecahedron model [2], and it has also been suggested that they rule out the possibility that we live in a small universe, since for the majority of detectable topologies we should expect antipodal or almost antipodal matched circles. In particular, it is argued that this claim is exact in all Euclidean manifolds with the only exception of the Hantzsche–Wendt manifold (\mathcal{G}_6 in Wolf’s notation [8]).

The purpose of this letter is twofold. First, it is shown how to use prior estimates on the local shape of the universe to reduce the region of the full parameter space in a way that the search for matched circles might become practical. In fact, it is shown how to calculate the unique point in the parameter space representing a pair of matched circles corresponding to a given isometry g (and its inverse). As a consequence, given some fine estimates of the covering group Γ of the present spatial section of our Universe, we may be able to confine, in a very effective way, the region of the parameter space in which to perform the searches for circles in the sky. This is the first important step towards the development of efficient strategies to look for these matched circles. Moreover, once such pairs of matched circles had been identified, it is a simple matter to use its location in the parameter space to determine with greater precision the topology of our Universe.

Second, it emerges from the calculations that we should not expect (nearly) antipodal matched circles from the majority of detectable topologies. In particular, any Euclidean topology, with the exception of the torus, might generate pairs of circles that are not even nearly antipodal, provided the observer lies out of the axis of rotation of the isometry that gives rise to the pair of circles. This result might be generalized to the spherical case, for which work is in progress.

The main motivation for this work is the suspicion that the alignment of the quadrupole and the octopole moments of CMB anisotropies observed by the satellite WMAP, together with the *anomalous* low value of the quadrupole moment, is the topological signature we should expect from a generic topology in a nearly flat universe, even if its size is slightly

¹These parameters are the center of each circle as a point in the sphere of last scattering (four parameters), the angular radius of both circles (one parameter), and the relative phase between them (one parameter).

larger than the horizon radius. Moreover, as has been shown in [9], if topology is detectable in a very nearly flat universe, the observable isometries will behave nearly as translations. If we locally approximate a nearly flat constant curvature space M with Euclidean space, the smallest isometries of the covering group of M will behave as isometries in Euclidean space. Since these isometries are not translations, they must behave as screw motions, thus an appropriate model to get a feeling of what to expect observationally in a nearly flat universe with detectable topology is a *twisted* cylinder.

Thus, let us begin by briefly describing the geometry of twisted cylinders. An isometry in Euclidean 3-space can always be written as (A, \mathbf{a}) , where \mathbf{a} is a vector and A is an orthogonal transformation, and its action on Euclidean space is given by

$$(A, \mathbf{a}) : \mathbf{x} \mapsto A\mathbf{x} + \mathbf{a} , \quad (1)$$

for any point \mathbf{x} . The generator of the covering group of a twisted cylinder is a screw motion, i.e. an isometry where its orthogonal part is a rotation and its translational part has a component parallel to the axis of rotation [8]. Thus we can always choose the origin and align the axis of rotation with the z -axis to write

$$A = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2)$$

for the orthogonal part, and

$$\mathbf{a} = (0, 0, L) \quad (3)$$

for the translational part of the generator $g = (A, \mathbf{a})$.

This is what is usually done when studying the mathematics of Euclidean manifolds, since it simplifies calculations. However, in cosmological applications this amounts to assume that the observer lies on the axis of rotation, which is a very unnatural assumption. In order to consider the arbitrariness of the position of the observer inside space, we parallel transport the axis of rotation, along the positive x -axis, a distance ρ from the origin which remains to be the observer's position. Thus the generator of the twisted cylinder is now $g = (A, \mathbf{b})$, with translational part given by

$$\mathbf{b} = \rho(1 - \cos \alpha) \hat{\mathbf{e}}_x - \rho \sin \alpha \hat{\mathbf{e}}_y + L \hat{\mathbf{e}}_z . \quad (4)$$

The pair of matched circles related by the generator $g = (A, \mathbf{b})$ are the intersections of the sphere of last scattering with its images under the isometries g and g^{-1} respectively, and the centers of these images are located at $g\mathbf{0} = \mathbf{b}$ and $g^{-1}\mathbf{0} = -A^{-1}\mathbf{b}$. Thus the angular positions of the centers of the matched circles are

$$\mathbf{n}_1 = \frac{\mathbf{b}}{|\mathbf{b}|} \quad \text{and} \quad \mathbf{n}_2 = -\frac{A^{-1}\mathbf{b}}{|\mathbf{b}|} = -A^{-1}\mathbf{n}_1 . \quad (5)$$

There are four parameters we can determine using (2) and (4). The angle σ between \mathbf{n}_1 and the axis of rotation, the angle μ between the centers of the pair of matched circles, the angular size ν of both matched circles, and the phase-shift ϕ . It turns out that only three of them are independent, as should be expected since a screw motion has only three free parameters (ρ, L, α) .

We easily obtain

$$\cos \mu = \mathbf{n}_1 \cdot \mathbf{n}_2 = -\frac{1 + \tan^2 \sigma \cos \alpha}{1 + \tan^2 \sigma} \quad (6)$$

for the angular separation between both directions \mathbf{n}_1 and \mathbf{n}_2 , while σ is given by

$$\tan \sigma = \frac{\rho}{L} \sqrt{2(1 - \cos \alpha)} . \quad (7)$$

One can see from (6) and (7) that the matched circles will be antipodal only when the observer is on the axis of rotation ($\rho = 0$), or the isometry is a translation ($\alpha = 0$). In particular, this shows that in a universe with any topology \mathcal{G}_2 – \mathcal{G}_6 , if a screw motion of the covering group generates a pair of matched circles, they will not necessarily appear nearly antipodal to an observer located off the axis of rotation. As an example consider an observer in a \mathcal{G}_4 universe located at a distance $\rho = L/2$ from the axis of rotation of the generator screw motion ($\alpha = \pi/2$). From (6) and (7) it follows that $\mu \approx 132^\circ$.

Next, to compute the angular size of these circles, let R_{LSS} be the radius of the sphere of last scattering. Simple geometry shows that, since $|\mathbf{b}|$ is the distance between the two centers of the spheres whose intersections generate one of the matched circles, the angular size of this intersection is

$$\cos \nu = \frac{|\mathbf{b}|}{2R_{LSS}} = \frac{L}{2R_{LSS} \cos \sigma} . \quad (8)$$

The computation of the phase-shift between the matched circles (the last parameter we wish to constrain) is more involved. First we need to have an operational definition of this quantity. This is simply accomplished if we realize that there is a great circle that passes through the centers, \mathbf{n}_1 and \mathbf{n}_2 , of the matched circles. Orient this great circle such that it passes first through \mathbf{n}_2 , and then through \mathbf{n}_1 along the shortest path, and let \mathbf{v}_2 , \mathbf{u}_2 , \mathbf{u}_1 and \mathbf{v}_1 be the intersections of the great circle with the matched ones following this orientation. If there were no phase-shift, then we would have $g(R_{LSS}\mathbf{u}_2) = R_{LSS}\mathbf{u}_1$ and $g(R_{LSS}\mathbf{v}_2) = R_{LSS}\mathbf{v}_1$. Hence we define the phase shift as the rotation angle, around the normal of the sphere at \mathbf{n}_1 , that takes \mathbf{u}_1 to $\hat{\mathbf{u}}_2 = g(R_{LSS}\mathbf{u}_2)/R_{LSS}$, positive if the shift is counterclockwise, and negative otherwise.

In order to use this operational definition to compute the phase-shift, recall first that the great circle passing through \mathbf{n}_2 and \mathbf{n}_1 , with the required orientation, is given by

$$\mathbf{n}(t) = \frac{1}{\sin \mu} [\mathbf{n}_2 \sin(\mu - t) + \mathbf{n}_1 \sin t] , \quad (9)$$

where t is the angular distance between $\mathbf{n}(t)$ and \mathbf{n}_2 . Thus we have

$$\begin{aligned} \mathbf{u}_1 &= \frac{1}{\sin \mu} [\mathbf{n}_2 \sin \nu + \mathbf{n}_1 \sin(\mu - \nu)] , \\ \mathbf{u}_2 &= \frac{1}{\sin \mu} [\mathbf{n}_2 \sin(\mu - \nu) + \mathbf{n}_1 \sin \nu] , \quad \text{and} \\ \hat{\mathbf{u}}_2 &= \frac{1}{\sin \mu} [A\mathbf{n}_2 \sin(\mu - \nu) + A\mathbf{n}_1 \sin \nu] + \frac{\mathbf{b}}{R_{LSS}} , \end{aligned} \quad (10)$$

Writing the positions of \mathbf{u}_1 and $\hat{\mathbf{u}}_2$ with respect to their projections to the axis \mathbf{n}_1 , as $\mathbf{w}_1 = \mathbf{u}_1 - \mathbf{n}_1 \cos \nu$ and $\mathbf{w}_2 = \hat{\mathbf{u}}_2 - \mathbf{n}_1 \cos \nu$, enables us to express easily the phase-shift as

$$\cos \phi = \frac{\mathbf{w}_1 \cdot \mathbf{w}_2}{\sin^2 \nu} , \quad (11)$$

since $|\mathbf{w}_1| = |\mathbf{w}_2| = \sin \nu$. After a somewhat lengthy calculation one arrives at

$$\cos \phi = \frac{2(1 + \cos \alpha)}{1 - \cos \mu} - 1 \quad (12)$$

for the phase-shift. It is easily seen that when the observer is on the axis of rotation ($\rho = 0$), the shift equals α ;² while when the isometry is a translation ($\alpha = 0$), the shift vanishes. In general, however, the shift depends on the three parameters (ρ, L, α) , but only through the values of μ and α , thus it is not an independent parameter. In fact, for a given pair (σ, μ) , one can easily compute ϕ , since α is readily obtained from (6).

Summarizing, given estimates of the parameters (ρ, L, α) , and having determined an estimate of the axis of rotation of the screw motion, one can perform searches for pairs of circles both with centers at an angular distance σ of this axis and separation μ between them, given by (7) and (6) respectively. The phase-shift between the circles is fixed by these two parameters. Moreover, we can also limit the search of matched circles to only those with angular size ν given by (8). We have, thus, constrained three out of the six parameters needed to locate a pair of matched circles. The three missing parameters are the position (θ, φ) of the axis of rotation of the screw motion and the azimuthal angle λ of the center of one of the matched circles.

Under the hypothesis that the alignment of the quadrupole and the octopole moments is due to the topology of space, the position of the axis of rotation might be estimated from this alignment. Work is in progress in this direction. Finally, only the angle λ remains totally unconstrained.

Interestingly, a consequent precise identification of a pair of matched circles will allow, reciprocally, to determine with greater precision the same topological parameters with which we started, together with our position and orientation in the Universe.

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²We infer from this that, in the general case, both angles α and ϕ have the same sign.

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